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Two-dimensional equations for electromagnetic waves in multi-layered thin dielectric films

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Abstract

Two-dimensional equations for electromagnetic fields in a multi-layered thin dielectric film are derived from the three-dimensional equations of electrodynamics by expanding the vector potential of the electromagnetic fields into trigonometric series expansions of the film thickness coordinate. The lower order equations are examined. It is shown that they can describe certain long waves in the film. The equations are useful for modeling thin film devices.

Keywords: Plate; Electromagnetic

1. Introduction

For mechanical waves in a film or plate it has been known for a long time that two-dimensional equations can be derived from the three-dimensional theory of elasticity by expanding the mechanical displacement vector into power or trigonometric series expansions of the plate thickness coordinate (Mindlin, 1955). According to Mindlin (1955), this procedure can be traced back to Cauchy and Poisson. The resulting two-dimensional plate equations are approximate in nature. Since they are much simpler than the three-dimensional equations, they often allow theoretical analyses of waves propagating in a plate (Mindlin, 1955), and waves in a plate on a substrate (Tiersten, 1969). The literature on elastic plates is numerous. For electromagnetic waves in a single-layered dielectric film, two-dimensional equations were derived in a similar manner in Lee and Yang (1993a) and Lee and Yu (1994), and were used in the analysis of plate waveguides and resonators (Lee and Yang, 1993b; Lee et al., 1994, 1996). Two-dimensional equations for

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coupled electromagnetic and mechanical fields in a plate were also developed (Eringen, 1989; Altay and Dokmeci, submitted for publication). The two-dimensional equations in Lee and Yang (1993a), Lee and Yu (1994), Eringen (1989) and Altay and Dokmeci (submitted for publication) are all for single-layered films. Multi-layered films are common structures for microwave devices (Marcuse, 1989). The recent development of superlattice offers more possibilities of multi-layered devices (Christen et al., 2003). In this paper we derive two-dimensional equations for electromagnetic fields in a multi-layered thin dielectric film. The three-dimensional equations are summarized in Section 2. Two-dimensional equations are obtained in Section 3, and are reduced to a few special cases in Section 4. To examine the accuracy of the two-dimensional equations, in Section 5 we compare solutions for a few waves in a three-layered dielectric waveguide from the two-dimensional equations with solutions for the same waves from the three-dimensional equations. Finally, some conclusions are drawn in Section 6.

2. Three-dimensional equations

The three-dimensional equations of electrodynamics are (Panofsky, 1955)

$$\varepsilon_{ijk}E_{k,j} = -\dot{B}_i, \quad \varepsilon_{ijk}H_{k,j} = \dot{D}_i + J_i, \quad B_{i,i} = 0, \quad D_{i,i} = \rho, \tag{1}$$

where **E** is the electric field, **D** is the electric displacement, **B** is the magnetic induction, **H** is the magnetic field, **J** is the free current density, and ρ is the free charge density. The summation convention for repeated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index are used. A superimposed dot represents differentiation with respect to time t. ε_{ijk} is the permutation tensor. The equations in (1) are accompanied by the following constitutive equations describing behaviors of a specific material:

$$D_i = \varepsilon_{ij}E_j, \qquad B_i = \mu_{ij}H_j, \tag{2}$$

where ε_{ij} is the electric permittivity, and μ_{ij} is the magnetic permeability. With the introduction of a vector potential **A** and a scalar potential ϕ by (Panofsky, 1955)

$$E_k = -\phi_{,k} - \dot{A}_k, \qquad B_k = \varepsilon_{kij}A_{j,i}. \tag{3}$$

Eq. $(1)_{1,3}$ are identically satisfied. $(1)_{2,4}$ can be written as equations in terms of the potentials. Consider a finite dielectric body occupying a region V (see Fig. 1). The boundary surface of V is denoted by S, with a unit exterior normal **n**. For boundary conditions we consider the following partitions of S (see Fig. 1):

$$\begin{aligned} &S_\phi \cup S_D = S_A \cup S_H = S, \\ &S_\phi \cap S_D = S_A \cap S_H = 0. \end{aligned} \tag{4}$$

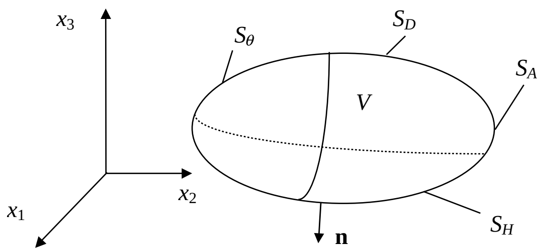

Fig. 1. A dielectric body and partitions of its boundary.

On S we may prescribe

$$\begin{aligned} &\phi = \bar{\phi}, \quad \text{on } S_\phi, \\ &D_i n_i + \bar{d} = 0, \quad \text{on } S_D, \\ &\varepsilon_{ijk} n_j A_k = \bar{a}_i, \quad \text{on } S_A, \\ &\varepsilon_{ijk} n_j H_k = \bar{h}_i, \quad \text{on } S_H, \end{aligned} \tag{5}$$

where $\bar{\phi}, \bar{d}, \bar{a}_i$ and $\bar{h}_i$ are known boundary data. Consider the following variational functional (Nelson, 1979; Lee, 1991; Yang, 1991):

$$\Pi(\mathbf{A}, \phi) = \int_{t_0}^{t_1} \mathrm{d}t \int_V \left[\frac{1}{2}(\varepsilon_{ij} E_i E_j - \mu_{ij}^{-1} B_i B_j) + J_i A_i - \rho\phi\right] \mathrm{d}V - \int_{t_0}^{t_1} \mathrm{d}t \int_{S_D} \bar{d}\phi \, \mathrm{d}S - \int_{t_0}^{t_1} \mathrm{d}t \int_{S_H} \bar{h}_i A_i \, \mathrm{d}S \tag{6}$$

with admissible functions satisfying

$$\begin{aligned} &\phi = \bar{\phi}, \quad \text{on } S_\phi, \\ &\varepsilon_{ijk} n_j A_k = \bar{a}_i, \quad \text{on } S_A, \\ &A_i(\mathbf{x}, t_0) = A_i^0, \quad A_i(\mathbf{x}, t_1) = A_i^1, \quad \text{in } V, \end{aligned} \tag{7}$$

where A_i^0 and A_i^1 are prescribed data at t_0 and t_1. Then

$$\begin{aligned} \delta\Pi = &\int_{t_0}^{t_1} \mathrm{d}t \int_V [(D_{i,i} - \rho)\delta\phi - (\varepsilon_{ijk} H_{k,j} - \dot{D}_i - J_i)\delta A_i] \, \mathrm{d}V - \int_{t_0}^{t_1} \mathrm{d}t \int_{S_D} (D_i n_i + \bar{d})\delta\phi \, \mathrm{d}S \\ &+ \int_{t_0}^{t_1} \mathrm{d}t \int_{S_H} (\varepsilon_{ijk} n_j H_k - \bar{h}_i)\delta A_i \, \mathrm{d}S. \end{aligned} \tag{8}$$

Therefore the stationary condition of (6) yields $(1)_{2,4}$ and $(5)_{2,4}$. In the problems we are interested in, $\mathbf{J}$ and ρ are both zero. For waves in a source free region, a vector potential $\mathbf{A}$ alone is sufficient (Lee and Yang, 1993). In this case $D_{i,i} = 0$ is essentially implied by $\dot{D}_i = \varepsilon_{ijk} H_{k,j}$.

3. Derivation of two-dimensional equations

Consider an N-layered plate of total thickness $2h$ (see Fig. 2). x_2 is the thickness coordinate. The x_3 and x_1 axes are in the middle surface. The two major surfaces and the $N-1$ interfaces are sequentially determined by $x_2 = -h = h_0, h_1, \ldots, h_{N-1}$, and $h_N = h$.

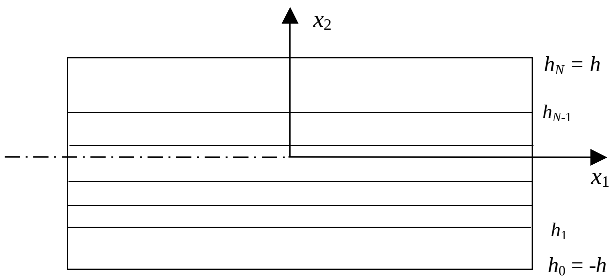

Fig. 2. A multi-layered thin film.

3.1. Potential expansions

We begin with the following expansion of the vector potential (Lee and Yang, 1993a; Lee and Yu, 1994):

$$
\begin{aligned}
A_a(x_1,x_2,x_3,t) &= \sum_{n=0}^{\infty} A_a^{(n)}(x_1,x_3,t)\cos\frac{n\pi}{2}(1-\psi), \quad a=1,3,\\
A_2(x_1,x_2,x_3,t) &= \sum_{n=0}^{\infty} A_2^{(n)}(x_1,x_3,t)\sin\frac{(n+1)\pi}{2}(1-\psi), \quad \phi=0,
\end{aligned}
\tag{9}
$$

where

$$
\psi = \frac{x_2}{h}. \tag{10}
$$

We want to derive two-dimensional equations for $A_i^{(n)}$. Substituting (9) into (3), we can write

$$
\begin{aligned}
E_a &= \sum_{n=0}^{\infty} E_a^{(n)}\cos\frac{n\pi}{2}(1-\psi),\\
E_2 &= \sum_{n=0}^{\infty} E_2^{(n)}\sin\frac{(n+1)\pi}{2}(1-\psi),\\
B_a &= \sum_{n=0}^{\infty} B_a^{(n)}\sin\frac{(n+1)\pi}{2}(1-\psi),\\
B_2 &= \sum_{n=0}^{\infty} B_2^{(n)}\cos\frac{n\pi}{2}(1-\psi),
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
E_i^{(n)} &= -\dot{A}_i^{(n)},\\
B_1^{(n)} &= \frac{(n+1)\pi}{2h}A_3^{(n+1)} - A_{2,3}^{(n)},\\
B_2^{(n)} &= A_{1,3}^{(n)} - A_{3,1}^{(n)},\\
B_3^{(n)} &= A_{2,1}^{(n)} - \frac{(n+1)\pi}{2h}A_1^{(n+1)}.
\end{aligned}
\tag{12}
$$

3.2. Field equations

Let the two-dimensional region occupied by the middle plane of a finite film in the x_3-x_1 plane be A (see Fig. 3). The volume integrals in (8) can be written as

$$
\begin{aligned}
\delta\Pi &= \int_{t_0}^{t_1}\mathrm{d}t\int_V -(\varepsilon_{ijk}H_{k,j}-\dot{D}_i)\delta A_i\,\mathrm{d}V\\
&= \int_{t_0}^{t_1}\mathrm{d}t\int_A \mathrm{d}x_1\,\mathrm{d}x_3\sum_{I=1}^{N}\int_{h_{I-1}}^{h_I}[-(H_{3,2}-H_{2,3}-\dot{D}_1)\delta A_1-(H_{1,3}-H_{3,1}-\dot{D}_2)\delta A_2\\
&\quad -(H_{2,1}-H_{1,2}-\dot{D}_3)\delta A_3]\,\mathrm{d}x_2.
\end{aligned}
\tag{13}
$$

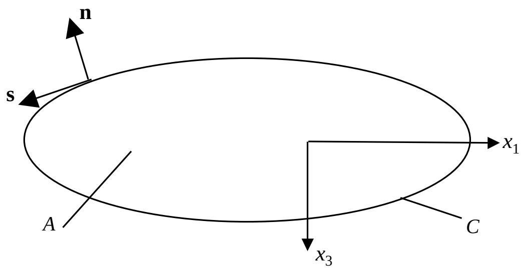

Fig. 3. A finite region in the middle surface of the film and its boundary.

Substitution of (9) into (13) yields

$$\begin{aligned} &-\frac{n\pi}{2h}H_3^{(n-1)} - H_{2,3}^{(n)} + \frac{1}{h}\mathsf{H}_3^{(n)} = \dot{D}_1^{(n)}, \\ &H_{1,3}^{(n)} - H_{3,1}^{(n)} = \dot{D}_2^{(n)}, \\ &H_{2,1}^{(n)} + \frac{n\pi}{2h}H_1^{(n-1)} - \frac{1}{h}\mathsf{H}_1^{(n)} = \dot{D}_3^{(n)}, \end{aligned} \tag{14}$$

where the resultant of the **H** field and the surface terms are defined by

$$\begin{aligned} &H_a^{(n)} = \int_{-1}^{1} H_a \sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi, \qquad H_2^{(n)} = \int_{-1}^{1} H_2 \cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi, \\ &D_a^{(n)} = \int_{-1}^{1} D_a \cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi, \qquad D_2^{(n)} = \int_{-1}^{1} D_2 \sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi, \\ &\mathsf{H}_a^{(n)} = H_a(h) - (-1)^n H_a(-h). \end{aligned} \tag{15}$$

Note that because of the use of the variational formulation and $\phi = 0$, there are no two-dimensional equations corresponding to $D_{i,i} = 0$. This is an important simplification compared to the equations in Lee and Yang (1993a) and Lee and Yu (1994) which were derived without using the variational formulation. Since a multi-layered film is a body with piecewise constant material parameters which do not have derivatives across an interface, the integrations in the variational formulation have to be performed layer by layer. Interface continuity conditions on the potentials are guaranteed by (9). Continuity of tangential **H** and normal **D** are part of the stationary conditions of the variational procedure. These conditions can only be considered as being satisfied approximately by two-dimensional solutions.

3.3. Constitutive relations

For the Ith layer, we write the constitutive relations as

$$D_i = \varepsilon_{ij}^I E_j, \qquad H_i = v_{ij}^I B_j, \tag{16}$$

where v_{ij}^I is the inverse of μ_{ij}^I. Substituting (16) and (11) into (15), we obtain the following two-dimensional constitutive relations (see Appendix A):

$$\begin{aligned} D_i^{(n)} &= \sum_m M_{ij}^{(m,n)} E_j^{(m)}, \\ H_i^{(n)} &= \sum_m N_{ij}^{(m,n)} B_j^{(m)}. \end{aligned} \tag{17}$$

Eq. (17) are relations among the two-dimensional fields. Their coefficients depend on the material constants and geometry of the layers.

3.4. Boundary conditions

In summary, we have obtained the field-potential relations (12), field equations (14), and constitutive relations (17). With successive substitutions, (12) can be written as equations for potentials of various orders. Let the boundary curve of the two-dimensional region A occupied by the middle surface of a finite film be C which has a unit outward normal n_a, $a = 1, 3$. Then a unit tangent s_a can be determined by (see Fig. 3)

$$\begin{aligned} &\mathbf{s} = \mathbf{e}_2 \times \mathbf{n}, \qquad s_i = \varepsilon_{ijk}\delta_{j2}n_k, \\ &s_1 = n_3, \quad s_2 = 0, \quad s_3 = -n_1, \end{aligned} \tag{18}$$

where $\mathbf{e}_2$ is the unit vector along x_2. We also introduce an $\overline{H}_i$ such that

$$\bar{h}_i = \varepsilon_{ijk}n_j\overline{H}_k. \tag{19}$$

Then the two-dimensional boundary conditions can be determined from the last term in (8). It can be concluded (see Appendix B) that on the boundary curve of a two-dimensional domain we may prescribe

$$\begin{aligned} &H_2 \quad \text{or} \quad \mathbf{A}\cdot\mathbf{s} \\ &\text{and} \quad \mathbf{H}\cdot\mathbf{s} \quad \text{or} \quad A_2. \end{aligned} \tag{20}$$

The determination of the form of the boundary conditions is another advantage of using the variational formulation.

4. Special cases

The two-dimensional equations obtained in the previous section are rather general. In this section we reduce them to a few special cases.

4.1. An isotropic film

When every layer is isotropic, we have

$$\varepsilon^I_{ij} = \varepsilon^I\delta_{ij}, \qquad v^I_{ij} = v^I\delta_{ij}, \tag{21}$$

which implies

$$\begin{aligned} &M^{(m,n)}_{a2} = 0, \quad M^{(m,n)}_{2a} = 0, \quad M^{(m,n)}_{13} = M^{(m,n)}_{31} = 0, \\ &M^{(m,n)}_{11} = M^{(m,n)}_{33} = \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\varepsilon^I\cos\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi, \\ &M^{(m,n)}_{22} = \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\varepsilon^I\sin\frac{(m+1)\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi, \end{aligned} \tag{22}$$

$$\begin{aligned} &N^{(m,n)}_{a2} = 0, \quad N^{(m,n)}_{2a} = 0, \quad N^{(m,n)}_{13} = N^{(m,n)}_{31} = 0, \\ &N^{(m,n)}_{11} = N^{(m,n)}_{33} = \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}v^I\sin\frac{(m+1)\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi, \\ &N^{(m,n)}_{22} = \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}v^I\cos\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi. \end{aligned} \tag{23}$$

Then the constitutive relations can be written as

$$\begin{aligned} D_a^{(n)} &= \sum_m M_{11}^{(m,n)} E_a^{(m)}, \qquad D_2^{(n)} = \sum_m M_{22}^{(m,n)} E_2^{(m)}, \\ H_a^{(n)} &= \sum_m N_{11}^{(m,n)} B_a^{(m)}, \qquad H_2^{(n)} = \sum_m N_{22}^{(m,n)} B_2^{(m)}. \end{aligned} \tag{24}$$

4.2. A single-layered film

When the film is of one-layer only, the constitutive relations reduce to

$$\begin{aligned} H_a^{(n)} &= v_{ab} B_b^{(n)} + v_{a2} \sum_m \gamma_{(n+1)m} B_2^{(m)}, \\ H_2^{(n)} &= v_{2b} \sum_m \gamma_{(m+1)n} B_b^{(m)} + v_{22}(1+\delta_{n0}) B_2^{(n)}, \\ D_a^{(n)} &= \varepsilon_{ab}(1+\delta_{n0}) E_b^{(n)} + \varepsilon_{a2} \sum_m \gamma_{(m+1)n} E_2^{(m)}, \\ D_2^{(n)} &= \varepsilon_{2b} \sum_m \gamma_{(n+1)m} E_b^{(m)} + \varepsilon_{22} E_2^{(n)}, \end{aligned} \tag{25}$$

where δ_{mn} is the Chronecker delta, and

$$\gamma_{mn} = \int_{-1}^{1} \sin\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,d\psi = \begin{cases} 0, & m+n \text{ even}, \\ \dfrac{4m}{(m^2-n^2)\pi}, & m+n \text{ odd}. \end{cases} \tag{26}$$

Eq. (25) are the equations derived by Lee and Yu (1994).

4.3. A single-layered isotropic film

If a film is isotropic and single layered, (25) further reduces to

$$\begin{aligned} D_a^{(n)} &= \varepsilon(1+\delta_{n0}) E_a^{(n)}, \qquad D_2^{(n)} = \varepsilon E_2^{(n)}, \\ H_a^{(n)} &= v B_a^{(n)}, \qquad H_2^{(n)} = v(1+\delta_{n0}) B_2^{(n)}, \end{aligned} \tag{27}$$

which are the equations derived by Lee and Yang (1993).

5. Applications to waveguides

Special cases of the two-dimensional equations obtained have been examined for waves in single-layered isotropic and anisotropic films (Lee and Yang, 1993a; Lee and Yu, 1994). In these special cases the dispersion curves of waves described by the two-dimensional equations show good agreement with the dispersion curves of the same waves when described by the three-dimensional equations. To examine the more general case of multi-layered films we consider waves propagating in a three-layered isotropic waveguide (see Fig. 4). In applications symmetric waveguides are used most often. In isotropic waveguides waves can be separated into transverse electric (TE) and transverse magnetic (TM) waves. They will be analyzed separately, both from the three- and two-dimensional equations for comparison. We study straight-crested waves independent of x_3.

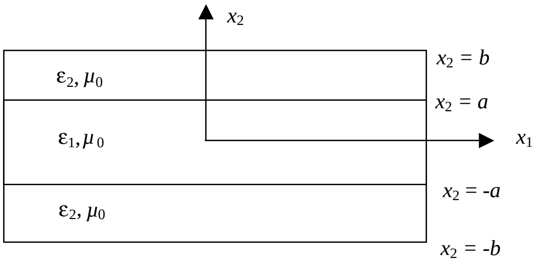

Fig. 4. A three-layered, symmetric and isotropic film.

5.1. TE waves from the three-dimensional equations

TE waves can be described by

$$A_1 = A_2 = 0, \qquad A_3 = A_3(x_1, t). \tag{28}$$

From (1)–(3) the governing equations are

$$A_{3,11} + A_{3,22} = \frac{1}{c^2}\ddot{A}_3, \quad \frac{1}{c^2} = \varepsilon\mu_0, \tag{29}$$

where μ_0 is the magnetic permeability of free space. The following A_3 represents a wave solution to (29):

$$\begin{aligned} &A_3 = (C\cos\eta x_2 + G\sin\eta x_2)\exp[\mathrm{i}(\xi x_1 - \omega t)], \\ &\eta^2 = \frac{\omega^2}{c^2} - \xi^2 = \xi^2\left(\frac{v^2}{c^2} - 1\right), \quad v^2 = \frac{\omega^2}{\xi^2}, \end{aligned} \tag{30}$$

where C and G are undetermined constants. $(30)_1$ generates the following field components:

$$\begin{aligned} &E_3 = \mathrm{i}\omega(C\cos\eta x_2 + G\sin\eta x_2)\exp[\mathrm{i}(\xi x_1 - \omega t)], \\ &B_1 = \eta(-C\sin\eta x_2 + G\cos\eta x_2)\exp[\mathrm{i}(\xi x_1 - \omega t)], \\ &B_2 = -\mathrm{i}\xi(C\cos\eta x_2 + G\sin\eta x_2)\exp[\mathrm{i}(\xi x_1 - \omega t)]. \end{aligned} \tag{31}$$

Eqs. (28)–(31) apply to each layer of the waveguide in Fig. 4 when the corresponding material constants of the layer are used. Therefore we write the TE wave solutions in each layer as

$$\begin{aligned} &A_3 = (C_1\cos\eta_1 x_2 + G_1\sin\eta_1 x_2)\exp[\mathrm{i}(\xi x_1 - \omega t)], \quad |x_2| < a, \\ &A_3 = (C_2\cos\eta_2 x_2 + G_2\sin\eta_2 x_2)\exp[\mathrm{i}(\xi x_1 - \omega t)], \quad a < x_2 < b, \\ &A_3 = (C_3\cos\eta_2 x_2 + G_3\sin\eta_2 x_2)\exp[\mathrm{i}(\xi x_1 - \omega t)], \quad -b < x_2 < -a, \end{aligned} \tag{32}$$

where

$$\begin{aligned} &\eta_1^2 = \frac{\omega^2}{c_1^2} - \xi^2 = \xi^2\left(\frac{v^2}{c_1^2} - 1\right), \qquad c_1^2 = \frac{1}{\varepsilon_1\mu_0}, \\ &\eta_2^2 = \frac{\omega^2}{c_2^2} - \xi^2 = \xi^2\left(\frac{v^2}{c_2^2} - 1\right), \qquad c_2^2 = \frac{1}{\varepsilon_2\mu_0}. \end{aligned} \tag{33}$$

At the interfaces of $x_2 = \pm a$ we must impose the continuity of E_3, H_1 and B_2. The continuity of E_3 and B_2 are not independent. Electric fields may also exist in the free space surrounding the film. If the fields in the film can be well described by the two-dimensional equations, the two-dimensional equations can be used together with the three-dimensional equations for the fields in the free space in the manner of Tiersten

(1969). Therefore our main purpose is to examine waves in the film, for which we study the case of perfect magnetic walls (Lee and Yang, 1993) with vanishing tangential **H** and normal **D** at $x_2 = \pm b$. Then the continuity and boundary conditions require

$$\begin{aligned}
&C_1 \cos\eta_1 a + G_1 \sin\eta_1 a = C_2 \cos\eta_2 a + G_2 \sin\eta_2 a,\\
&C_1 \cos\eta_1 a - G_1 \sin\eta_1 a = C_3 \cos\eta_2 a - G_3 \sin\eta_2 a,\\
&\eta_1(-C_1 \sin\eta_1 a + G_1 \cos\eta_1 a) = \eta_2(-C_2 \sin\eta_2 a + G_2 \cos\eta_2 a),\\
&\eta_1(C_1 \sin\eta_1 a + G_1 \cos\eta_1 a) = \eta_2(C_3 \sin\eta_2 a + G_3 \cos\eta_2 a),\\
&-C_2 \sin\eta_2 b + G_2 \cos\eta_2 b = 0,\\
&C_3 \sin\eta_2 b + G_3 \cos\eta_2 b = 0.
\end{aligned} \tag{34}$$

For nontrivial solutions the determinant of the coefficient matrix has to vanish, which determines the dispersion relations of the waves

$$\begin{aligned}
&[\eta_1 \sin(\eta_1 a)\cos\eta_2(a-b) - \eta_2 \cos(\eta_1 a)\sin\eta_2(a-b)]\\
&\quad\times[\eta_1 \cos(\eta_1 a)\cos\eta_2(a-b) + \eta_2 \sin(\eta_1 a)\sin\eta_2(a-b)] = 0.
\end{aligned} \tag{35}$$

Eq. (35) has two factors, one for symmetric waves and the other for anti-symmetric waves. In terms of the wave speed v, (35) can be written as

$$\begin{aligned}
&\tan\xi a\left(\frac{v^2}{c_1^2}-1\right)^{1/2}\cot\xi(a-b)\left(\frac{v^2}{c_2^2}-1\right)^{1/2} = \left(\frac{v^2}{c_2^2}-1\right)^{1/2}\Big/\left(\frac{v^2}{c_1^2}-1\right)^{1/2},\\
&\cot\xi a\left(\frac{v^2}{c_1^2}-1\right)^{1/2}\cot\xi(a-b)\left(\frac{v^2}{c_2^2}-1\right)^{1/2} = -\left(\frac{v^2}{c_2^2}-1\right)^{1/2}\Big/\left(\frac{v^2}{c_1^2}-1\right)^{1/2}.
\end{aligned} \tag{36}$$

Clearly, (36) describes dispersive waves. When $a = b$, it reduces to the equations for the dispersion relations of TE waves in a single-layered film (Lee and Yang, 1993).

5.2. TM waves from the three-dimensional equations

TM waves are described by

$$A_1 = A_1(x_1, t), \quad A_2 = A_2(x_1, t), \quad A_3 = 0. \tag{37}$$

The governing equations are

$$A_{1,22} - A_{2,12} = \frac{1}{c^2}\ddot{A}_1, \qquad A_{2,11} - A_{1,12} = \frac{1}{c^2}\ddot{A}_2, \tag{38}$$

which are equivalent to

$$\begin{aligned}
&A_{1,11} + A_{1,22} = \frac{1}{c^2}\ddot{A}_1, \qquad A_{2,11} + A_{2,22} = \frac{1}{c^2}\ddot{A}_2,\\
&A_{1,1} + A_{2,2} = 0.
\end{aligned} \tag{39}$$

The following represents a wave solution to (39):

$$\begin{aligned}
&A_1 = \frac{i\eta}{\xi}(-C\sin\eta x_2 + G\cos\eta x_2)\exp[i(\xi x_1 - \omega t)],\\
&A_2 = (C\cos\eta x_2 + G\sin\eta x_2)\exp[i(\xi x_1 - \omega t)].
\end{aligned} \tag{40}$$

Eq. (40) generates the following field components:

$$\begin{aligned}
B_3 &= \frac{i\omega v}{c^2}(C\cos\eta x_2 + G\sin\eta x_2)\exp[i(\xi x_1 - \omega t)],\\
E_1 &= v\eta(C\sin\eta x_2 - G\cos\eta x_2)\exp[i(\xi x_1 - \omega t)],\\
E_2 &= \frac{c^2}{v}B_3.
\end{aligned} \tag{41}$$

Eqs. (37)–(41) apply to each layer of the waveguide in Fig. 4 when the corresponding material constants of the layer are used. Therefore we write the TM wave solutions in each layer as

$$\begin{aligned}
A_1 &= \frac{i\eta_1}{\xi}(-C_1\sin\eta_1 x_2 + G_1\cos\eta_1 x_2)\exp[i(\xi x_1 - \omega t)], \quad |x_2| < a,\\
A_2 &= (C_1\cos\eta_1 x_2 + G_1\sin\eta_1 x_2)\exp[i(\xi x_1 - \omega t)], \quad |x_2| < a,\\
A_1 &= \frac{i\eta_2}{\xi}(-C_2\sin\eta_2 x_2 + G_2\cos\eta_2 x_2)\exp[i(\xi x_1 - \omega t)], \quad a < x_2 < b,\\
A_2 &= (C_2\cos\eta_2 x_2 + G_2\sin\eta_2 x_2)\exp[i(\xi x_1 - \omega t)], \quad a < x_2 < b,\\
A_1 &= \frac{i\eta_2}{\xi}(-C_3\sin\eta_2 x_2 + G_3\cos\eta_2 x_2)\exp[i(\xi x_1 - \omega t)], \quad -b < x_2 < -a,\\
A_2 &= (C_3\cos\eta_2 x_2 + G_3\sin\eta_2 x_2)\exp[i(\xi x_1 - \omega t)], \quad -b < x_2 < -a.
\end{aligned} \tag{42}$$

At the interfaces of $x_2 = \pm a$ we must impose the continuity of H_3, E_1 and D_2. The continuity of H_3 and D_2 are not independent. Consider the case of perfect magnetic walls at $x_2 = \pm b$. Then the continuity and boundary conditions require

$$\begin{aligned}
&\frac{i\omega v}{c_1^2}(C_1\cos\eta_1 a + G_1\sin\eta_1 a) = \frac{i\omega v}{c_2^2}(C_2\cos\eta_2 a + G_2\sin\eta_2 a),\\
&\frac{i\omega v}{c_1^2}(C_1\cos\eta_1 a - G_1\sin\eta_1 a) = \frac{i\omega v}{c_2^2}(C_3\cos\eta_2 a - G_3\sin\eta_2 a),\\
&\eta_1(C_1\sin\eta_1 a - G_1\cos\eta_1 a) = \eta_2(C_2\sin\eta_2 a - G_2\cos\eta_2 a),\\
&\eta_1(-C_1\sin\eta_1 a - G_1\cos\eta_1 a) = \eta_2(-C_3\sin\eta_2 a - G\cos\eta_2 a),\\
&C_2\cos\eta_2 b + G_2\sin\eta_2 b = 0,\\
&C_3\cos\eta_2 b - G_3\sin\eta_2 b = 0.
\end{aligned} \tag{43}$$

For nontrivial solutions the determinant of the coefficient matrix has to vanish, which determines the dispersion relations of the waves

$$\begin{aligned}
-\frac{2}{c_1^4 c_2^4}\big\{&\big[c_1^2\eta_1\sin(\eta_1 a)\sin\eta_2(a-b) + c_2^2\eta_2\cos(\eta_1 a)\cos\eta_2(a-b)\big]\\
&\times\big[c_1^2\eta_1\cos(\eta_1 a)\sin\eta_2(a-b) - c_2^2\eta_2\sin(\eta_1 a)\cos\eta_2(a-b)\big]\big\} = 0.
\end{aligned} \tag{44}$$

In terms of v, (44) takes the following form:

$$\begin{aligned}
&\tan\xi a\left(\frac{v^2}{c_1^2}-1\right)^{1/2}\tan\xi(a-b)\left(\frac{v^2}{c_2^2}-1\right)^{1/2} = -c_2^2\left(\frac{v^2}{c_2^2}-1\right)^{1/2}\Big/ c_1^2\left(\frac{v^2}{c_1^2}-1\right)^{1/2},\\
&\cot\xi a\left(\frac{v^2}{c_1^2}-1\right)^{1/2}\tan\xi(a-b)\left(\frac{v^2}{c_2^2}-1\right)^{1/2} = c_2^2\left(\frac{v^2}{c_2^2}-1\right)^{1/2}\Big/ c_1^2\left(\frac{v^2}{c_1^2}-1\right)^{1/2},
\end{aligned} \tag{45}$$

which describes dispersive waves. When $a = b$, (45) reduces to the equations for the dispersion relations of TM waves in a single-layered film (Lee and Yang, 1993).

5.3. TE waves from uncoupled two-dimensional equations

Consider

$$A_1^{(n)} = A_2^{(n)} = 0, \qquad A_3^{(n)} = A_3^{(n)}(x_1, t). \tag{46}$$

Then the nontrivial field components are

$$E_3^{(n)} = -\dot{A}_3^{(n)}, \quad B_1^{(n-1)} = \frac{n\pi}{2h} A_3^{(n)}, \quad B_2^{(n)} = -A_{3,1}^{(n)}. \tag{47}$$

From the constitutive relations we obtain

$$\begin{aligned} D_3^{(n)} &= -M_{11}^{(n,n)} \dot{A}_3^{(n)}, \\ H_1^{(n-1)} &= N_{11}^{(n-1,n-1)} \frac{n\pi}{2h} A_3^{(n)}, \\ H_2^{(n)} &= -N_{22}^{(n,n)} A_{3,1}^{(n)}, \end{aligned} \tag{48}$$

where couplings among fields of different orders ($m \neq n$) have been neglected. Substituting (48) into $(14)_3$ gives

$$-N_{22}^{(n,n)} A_{3,11}^{(n)} + N_{11}^{(n-1,n-1)} \left(\frac{n\pi}{2h}\right)^2 A_3^{(n)} = -M_{11}^{(n,n)} \ddot{A}_3^{(n)}, \tag{49}$$

where the surface term $\mathsf{H}_a^{(n)}$ vanishes for a perfect magnetic wall. Substituting a wave solution $A_3^{(n)} = \exp[\mathrm{i}(\xi x_1 - \omega t)]$ into (49) gives the following dispersion relation:

$$M_{11}^{(n,n)} \omega^2 = N_{22}^{(n,n)} \xi^2 + N_{11}^{(n-1,n-1)} \left(\frac{n\pi}{2h}\right)^2. \tag{50}$$

For the three-layered plate in Fig. 4, the coefficients in (50) are given by

$$\begin{aligned} M_{11}^{(n,n)} &= \varepsilon_1 \left(1 - \frac{a}{b}\right) - \frac{\varepsilon_1}{n\pi} \cos(n\pi) \sin\left(n\pi \frac{a}{b}\right) + \varepsilon_2 \frac{a}{b} + \frac{\varepsilon_2}{n\pi} \cos(n\pi) \sin\left(n\pi \frac{a}{b}\right), \\ N_{11}^{(n,n)} &= \frac{1}{\mu_0}\left(1 - \frac{a}{b}\right) + \frac{1}{\mu_0 (n+1)\pi} \cos[(n+1)\pi] \sin\left[(n+1)\pi \frac{a}{b}\right] \\ &\quad + \frac{1}{\mu_0} \frac{a}{b} - \frac{1}{\mu_0 (n+1)\pi} \cos[(n+1)\pi] \sin\left[(n+1)\pi \frac{a}{b}\right], \\ N_{22}^{(n,n)} &= \frac{1}{\mu_0}\left(1 - \frac{a}{b}\right) - \frac{1}{\mu_0 n\pi} \cos(n\pi) \sin\left(n\pi \frac{a}{b}\right) + \frac{1}{\mu_0} \frac{a}{b} + \frac{1}{\mu_0 n\pi} \cos(n\pi) \sin\left(n\pi \frac{a}{b}\right). \end{aligned} \tag{51}$$

5.4. TM waves from uncoupled two-dimensional equations

Consider

$$A_1^{(n)} = A_1^{(n)}(x_1, t), \quad A_2^{(n-1)} = A_2^{(n-1)}(x_1, t), \quad A_3^{(n)} = 0. \tag{52}$$

Then the nontrivial field components are

$$\begin{aligned} E_1^{(n)} &= -\dot{A}_1^{(n)}, \qquad E_2^{(n-1)} = -\dot{A}_2^{(n-1)}, \\ B_3^{(n-1)} &= A_{2,1}^{(n-1)} - \frac{n\pi}{2h} A_1^{(n)}. \end{aligned} \tag{53}$$

From the constitutive relations we obtain

$$\begin{aligned} &D_1^{(n)} = -M_{11}^{(n,n)}\dot{A}_1^{(n)}, \qquad D_2^{(n-1)} = -M_{22}^{(n-1,n-1)}\dot{A}_2^{(n-1)}, \\ &H_3^{(n-1)} = N_{11}^{(n-1,n-1)}\left(A_{2,1}^{(n-1)} - \frac{n\pi}{2h}A_1^{(n)}\right), \end{aligned} \tag{54}$$

Substituting (54) into $(14)_{1,2}$ gives

$$\begin{aligned} &-\frac{n\pi}{2h}N_{11}^{(n-1,n-1)}\left(A_{2,1}^{(n-1)} - \frac{n\pi}{2h}A_1^{(n)}\right) = -M_{11}^{(n,n)}\ddot{A}_1^{(n)}, \\ &-N_{11}^{(n-1,n-1)}\left(A_{2,11}^{(n-1)} - \frac{n\pi}{2h}A_{1,1}^{(n)}\right) = -M_{22}^{(n-1,n-1)}\ddot{A}_2^{(n-1)}, \end{aligned} \tag{55}$$

where couplings among different orders have been neglected. From (55) we can obtain

$$-\left(\frac{n\pi}{2h}\right)^2\frac{N_{11}^{(n-1,n-1)}}{M_{11}^{(n,n)}}\left(A_{2,1}^{(n-1)} - \frac{n\pi}{2h}A_1^{(n)}\right) + \frac{N_{11}^{(n-1,n-1)}}{M_{22}^{(n-1,n-1)}}\left(A_{2,111}^{(n-1)} - \frac{n\pi}{2h}A_{1,11}^{(n)}\right) = \ddot{A}_{2,1}^{(n-1)} - \frac{n\pi}{2h}\ddot{A}_1^{(n)} \tag{56}$$

or

$$-\left(\frac{n\pi}{2h}\right)^2\frac{N_{11}^{(n-1,n-1)}}{M_{11}^{(n,n)}}B_3^{(n-1)} + \frac{N_{11}^{(n-1,n-1)}}{M_{22}^{(n-1,n-1)}}B_{3,11}^{(n-1)} = \ddot{B}_3^{(n-1)}. \tag{57}$$

Substituting $B_3^{(n-1)} = \exp[\mathrm{i}(\xi x_1 - \omega t)]$ into (57) gives the dispersion relations of TM waves

$$\omega^2 = \frac{N_{11}^{(n-1,n-1)}}{M_{22}^{(n-1,n-1)}}\xi^2 + \left(\frac{n\pi}{2h}\right)^2\frac{N_{11}^{(n-1,n-1)}}{M_{11}^{(n,n)}}, \tag{58}$$

where

$$\begin{aligned} M_{22}^{(n,n)} = {} & \varepsilon_1\left(1 - \frac{a}{b}\right) + \frac{\varepsilon_1}{(n+1)\pi}\cos[(n+1)\pi]\sin\left[(n+1)\pi\frac{a}{b}\right] \\ & + \varepsilon_2\frac{a}{b} - \frac{\varepsilon_2}{(n+1)\pi}\cos[(n+1)\pi]\sin\left[(n+1)\pi\frac{a}{b}\right]. \end{aligned} \tag{59}$$

5.5. TE waves from coupled two-dimensional equations

In applications the first few modes are used often. In the simple, uncoupled two-dimensional equations above, couplings among different orders of the two-dimensional equations are neglected. The modes these uncoupled equations describe are approximations of the corresponding three-dimensional modes. If couplings among different orders of the two-dimensional equations are included, better approximations of the three-dimensional modes can be expected. For example, consider

$$\begin{aligned} &A_1 = 0, \qquad A_2 = 0, \\ &A_3 = \sum_{n=0}^{4} A_3^{(n)}(x_1,t)\cos\frac{n\pi}{2}(1-\psi). \end{aligned} \tag{60}$$

Then

$$E_3^{(m)} = -\dot{A}_3^{(m)}(x_1,t), \quad B_1^{(m-1)} = \frac{m\pi}{2h}A_3^{(m)}(x_1,t), \quad B_2^{(m)} = -A_{3,1}^{(m)}(x_1,t). \tag{61}$$

From the constitutive relations we obtain

$$\begin{aligned} D_3^{(n)} &= -\sum_{m=0}^{4} M_{11}^{(m,n)} \dot{A}_3^{(m)}(x_1,t), \\ H_1^{(n-1)} &= \sum_{m=0}^{4} N_{11}^{(m-1,n-1)} \frac{n\pi}{2h} A_3^{(m)}(x_1,t), \\ H_2^{(n)} &= -\sum_{m=0}^{4} N_{22}^{(m,n)} A_{3,1}^{(m)}(x_1,t). \end{aligned} \tag{62}$$

Substituting (62) into $(14)_3$

$$-\sum_{m=0}^{4} N_{22}^{(m,n)} A_{3,11}^{(m)}(x_1,t) + \left(\frac{n\pi}{2h}\right)^2 \sum_{m=0}^{4} N_{11}^{(m-1,n-1)} A_3^{(m)}(x_1,t) = -\sum_{m=0}^{4} M_{11}^{(m,n)} \ddot{A}_3^{(m)}(x_1,t), \tag{63}$$

where $n = 0,1,2,3,4$. Consider the propagation of the following wave:

$$A_3^{(m)} = C^{(m)} \exp[\mathrm{i}(\xi x_1 - \omega t)], \tag{64}$$

where $m = 0,1,2,3,4$. $C^{(m)}$ is the wave amplitude. Substituting (64) into (63) gives the following linear equations for $C^{(m)}$:

$$\begin{aligned} &\sum_{m=0}^{4} \left[N_{22}^{(m,0)} \xi^2 - M_{11}^{(m,0)} \omega^2\right] C^{(m)} = 0, \\ &\sum_{m=0}^{4} \left[N_{22}^{(m,1)} \xi^2 + \left(\frac{\pi}{2h}\right)^2 N_{11}^{(m-1,0)} - M_{11}^{(m,1)} \omega^2\right] C^{(m)} = 0, \\ &\sum_{m=0}^{4} \left[N_{22}^{(m,2)} \xi^2 + \left(\frac{\pi}{h}\right)^2 N_{11}^{(m-1,1)} - M_{11}^{(m,2)} \omega^2\right] C^{(m)} = 0, \\ &\sum_{m=0}^{4} \left[N_{22}^{(m,3)} \xi^2 + \left(\frac{3\pi}{2h}\right)^2 N_{11}^{(m-1,2)} - M_{11}^{(m,3)} \omega^2\right] C^{(m)} = 0, \\ &\sum_{m=0}^{4} \left[N_{22}^{(m,4)} \xi^2 + \left(\frac{2\pi}{h}\right)^2 N_{11}^{(m-1,3)} - M_{11}^{(m,4)} \omega^2\right] C^{(m)} = 0. \end{aligned} \tag{65}$$

For nontrivial solutions the determinant of the coefficient matrix has to vanish, which gives the dispersion relations. In (65),

$$\begin{aligned} M_{11}^{(m,n)} &= \frac{2(\varepsilon_2 - \varepsilon_1)}{(m+n)\pi} \cos(m+n)\pi \sin(m+n)\pi \frac{a}{b} + \frac{2(\varepsilon_2 - \varepsilon_1)}{(m-n)\pi} \cos(m-n)\pi \sin(m-n)\pi \frac{a}{b}, \\ N_{11}^{(m,n)} &= \frac{2(v_2 - v_1)}{(m-n)\pi} \cos(m-n)\pi \sin(m-n)\pi \frac{b}{a} - \frac{2(v_2 - v_1)}{(m+n+2)\pi} \cos(m+n+2)\pi \sin(m+n+2)\pi \frac{b}{a}, \\ N_{22}^{(m,n)} &= \frac{2(v_2 - v_1)}{(m+n)\pi} \cos(m+n)\pi \sin(m+n)\pi \frac{a}{b} + \frac{2(v_2 - v_1)}{(m-n)\pi} \cos(m-n)\pi \sin(m-n)\pi \frac{a}{b}. \end{aligned} \tag{66}$$

5.6. Comparisons of two- and three-dimensional solutions

We compare (36) with (50) and (65) for TE waves, and (45) with (58) for TM waves. Dispersion relations in terms of the following dimensionless variables are calculated and plotted:

$$X = \xi \Big/ \left(\frac{\pi}{2h} \right), \quad \Omega = \omega \Big/ \left(c_0 \frac{\pi}{2h} \right), \quad c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \tag{67}$$

We are interested in long waves with a small wave number X, which are used more often in applications.

Fig. 5 shows the comparison of the first few branches of the dispersion curves for TE waves. The figure shows that the dispersion curves of the uncoupled two-dimensional equations agree qualitatively with those from the three-dimensional equations, but the two-dimensional equations and solutions are much simpler. Similar results can be seen in Fig. 6 for TM waves.

If couplings among different orders of the two-dimensional equations are considered, the dispersion curves of the two-dimensional equations approximate those of the three-dimensional equations better (See Fig. 7 for TE waves and its comparison with Fig. 5). The cutoff frequencies (frequencies for vanishing wave numbers) are still off. These cutoff frequencies usually can be adjusted by introducing correction factor(s) (Mindlin, 1955; Lee and Yang, 1993a; Lee and Yu, 1994), which needs to be done in specific cases and is not pursued here.

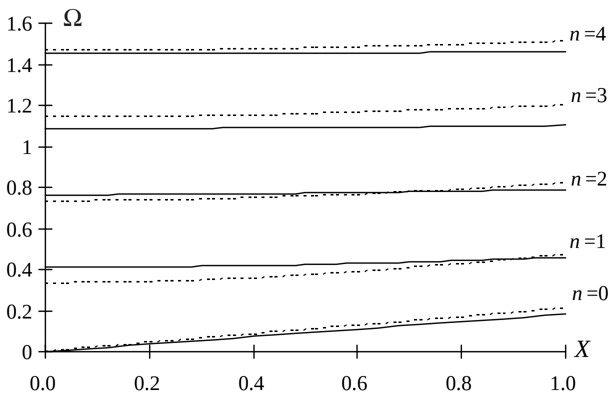

Fig. 5. Dispersion relations of TE waves. $b = 2a$, $\varepsilon_1 = 2\varepsilon_2$, solid lines: 3-D solutions, dotted lines: 2-D uncoupled solutions.

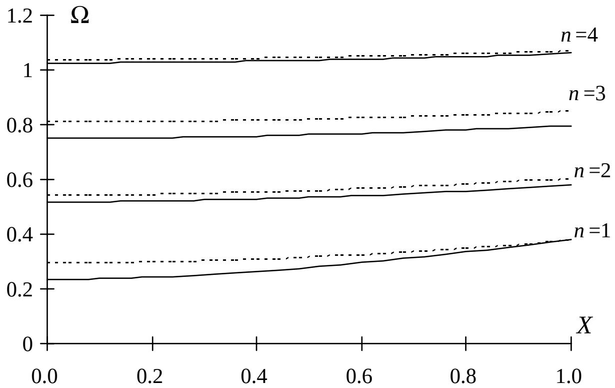

Fig. 6. Dispersion relations of TM waves. $b = 2a$, $\varepsilon_1 = 2\varepsilon_2$, solid lines: 3-D solutions, dotted lines: 2-D uncoupled solutions.

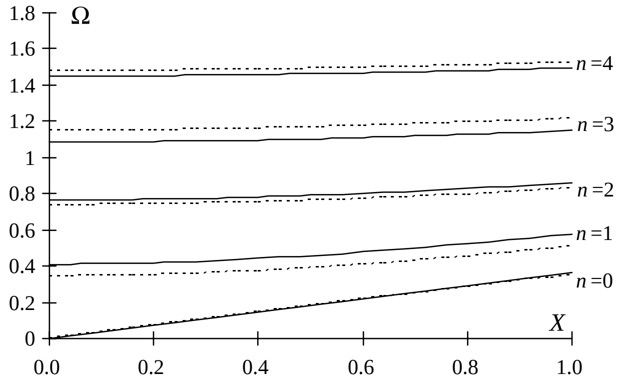

Fig. 7. Dispersion relations of TE waves. $b = 2a$, $\varepsilon_1 = 2\varepsilon_2$, solid lines: 3-D solutions, dotted lines: 2-D coupled solutions.

6. Conclusion

Two-dimensional equations for electromagnetic waves in layered dielectric plates are derived. The derivation differs from those in the literature by using a variational principle which results in a major simplification of the equations. The equations obtained can describe long waves in a multi-layered dielectric plate. They are simpler than the three-dimensional equations and can be used to study finite dielectric resonators, and surface waves guided by dielectric films.

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Appendix A. Derivation of two-dimensional constitutive relations

For $a, b = 1, 3$:

$$H_a^{(n)} = \int_{-1}^{1} H_a \sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi = \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}(v_{ab}^I B_b + v_{a2}^I B_2)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi$$
$$= \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\left[v_{ab}^I\sum_m B_b^{(m)}\sin\frac{(m+1)\pi}{2}(1-\psi) + v_{a2}^I\sum_m B_2^{(m)}\cos\frac{m\pi}{2}(1-\psi)\right]$$
$$\times\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi = \sum_m B_b^{(m)}\sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I} v_{ab}^I\sin\frac{(m+1)\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi$$
$$+\sum_m B_2^{(m)}\sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I} v_{a2}^I\cos\frac{m\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi$$
$$= \sum_m B_b^{(m)}N_{ab}^{(m,n)} + \sum_m B_2^{(m)}N_{a2}^{(m,n)} = \sum_m N_{aj}^{(m,n)}B_j^{(m)}, \qquad (A.1)$$

where $\psi_I = h_I/h$, and

$$N_{ab}^{(m,n)} = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} v_{ab}^I \sin\frac{(m+1)\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi = N_{ba}^{(m,n)} = N_{ab}^{(n,m)},$$
$$N_{a2}^{(m,n)} = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} v_{a2}^I \cos\frac{m\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi. \tag{A.2}$$

Similarly,

$$\begin{aligned} H_2^{(n)} &= \int_{-1}^{1} H_2 \cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} (v_{2b}^I B_b + v_{22}^I B_2)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &= \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} \left[v_{2b}^I \sum_m B_b^{(m)} \sin\frac{(m+1)\pi}{2}(1-\psi) + v_{22}^I \sum_m B_2^{(m)} \cos\frac{m\pi}{2}(1-\psi)\right]\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &= \sum_m B_b^{(m)} \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} v_{2b}^I \sin\frac{(m+1)\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &\quad + \sum_m B_2^{(m)} \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} v_{22}^I \cos\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &= \sum_m B_b^{(m)} N_{2b}^{(m,n)} + \sum_m B_2^{(m)} N_{22}^{(m,n)} = \sum_m N_{2j}^{(m,n)} B_j^{(m)}, \end{aligned} \tag{A.3}$$

where

$$N_{2b}^{(m,n)} = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} v_{2b}^I \sin\frac{(m+1)\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi = N_{b2}^{(n,m)},$$
$$N_{22}^{(m,n)} = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} v_{22}^I \cos\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi = N_{22}^{(n,m)}. \tag{A.4}$$

For the electric constitutive relations, we have

$$\begin{aligned} D_a^{(n)} &= \int_{-1}^{1} D_a \cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} (\varepsilon_{ab}^I E_b + \varepsilon_{a2}^I E_2)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &= \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} \left[\varepsilon_{ab}^I \sum_m E_b^{(m)} \cos\frac{m\pi}{2}(1-\psi) + \varepsilon_{a2}^I \sum_m E_2^{(m)} \sin\frac{(m+1)\pi}{2}(1-\psi)\right]\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &= \sum_m E_b^{(m)} \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} \varepsilon_{ab}^I \cos\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &\quad + \sum_m E_2^{(m)} \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} \varepsilon_{a2}^I \sin\frac{(m+1)\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi \\ &= \sum_m E_b^{(m)} M_{ab}^{(m,n)} + \sum_m E_2^{(m)} M_{a2}^{(m,n)} = \sum_m M_{aj}^{(m,n)} E_j^{(m)}, \end{aligned} \tag{A.5}$$

$$M_{ab}^{(m,n)} = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} \varepsilon_{ab}^I \cos\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi = M_{ba}^{(m,n)} = M_{ab}^{(n,m)},$$
$$M_{a2}^{(m,n)} = \sum_{I=1}^{N} \int_{\psi_{I-1}}^{\psi_I} \varepsilon_{a2}^I \sin\frac{(m+1)\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi, \tag{A.6}$$

$$
\begin{aligned}
D_2^{(n)} &= \int_{-1}^{1} D_2 \sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi = \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}(\varepsilon_{2b}^I E_b + \varepsilon_{22}^I E_2)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi \\
&= \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\left[\varepsilon_{2b}^I\sum_m E_b^{(m)}\cos\frac{m\pi}{2}(1-\psi) + \varepsilon_{22}^I\sum_m E_2^{(m)}\sin\frac{(m+1)\pi}{2}(1-\psi)\right]\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi \\
&= \sum_m E_b^{(m)}\sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\varepsilon_{2b}^I\cos\frac{(m\pi)}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi \\
&\quad + \sum_m E_2^{(m)}\sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\varepsilon_{22}^I\sin\frac{(m+1)\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi \\
&= \sum_m E_b^{(m)}M_{2b}^{(m,n)} + \sum_m E_2^{(m)}M_{22}^{(m,n)} = \sum_m M_{2j}^{(m,n)}E_j^{(m)},
\end{aligned}
\tag{A.7}
$$

where

$$
\begin{aligned}
M_{2b}^{(m,n)} &= \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\varepsilon_{2b}^I\cos\frac{m\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi = M_{b2}^{(n,m)}, \\
M_{22}^{(m,n)} &= \sum_{I=1}^{N}\int_{\psi_{I-1}}^{\psi_I}\varepsilon_{22}^I\sin\frac{(m+1)\pi}{2}(1-\psi)\sin\frac{(n+1)\pi}{2}(1-\psi)\,\mathrm{d}\psi = M_{22}^{(n,m)}.
\end{aligned}
\tag{A.8}
$$

Appendix B. Derivation of two-dimensional boundary conditions

$$
\begin{aligned}
&\int_{t_0}^{t_1}\mathrm{d}t\int_{S_H}\varepsilon_{ijk}n_j(H_k-\overline{H}_k)\delta A_i\,\mathrm{d}S \\
&\quad = \int_{t_0}^{t_1}\mathrm{d}t\int_{C_H}\mathrm{d}l\int_{-h}^{h}\mathrm{d}x_2[(n_2H_3 - n_2\overline{H}_3 - n_3H_2 + n_3\overline{H}_2)\delta A_1 \\
&\qquad + (n_3H_1 - n_3\overline{H}_1 - n_1H_3 + n_1\overline{H}_3)\delta A_2 \\
&\qquad + (n_1H_2 - n_1\overline{H}_2 - n_2H_1 + n_2\overline{H}_1)\delta A_3] \\
&\quad = \int_{t_0}^{t_1}\mathrm{d}t\int_{C_H}\mathrm{d}l\int_{-h}^{h}\mathrm{d}x_2[(-n_3H_2 + n_3\overline{H}_2)\delta A_1 \\
&\qquad + (n_3H_1 - n_3\overline{H}_1 - n_1H_3 + n_1\overline{H}_3)\delta A_2 + (n_1H_2 - n_1\overline{H}_2)\delta A_3] \\
&\quad = \int_{t_0}^{t_1}\mathrm{d}t\int_{C_H}\mathrm{d}l\int_{-h}^{h}\mathrm{d}x_2[(-s_1H_2 + s_1\overline{H}_2)\delta A_1 \\
&\qquad + (s_1H_1 - s_1\overline{H}_1 + s_3H_3 - s_3\overline{H}_3)\delta A_2 + (-s_3H_2 + s_3\overline{H}_2)\delta A_3] \\
&\quad = \int_{t_0}^{t_1}\mathrm{d}t\int_{C_H}\mathrm{d}l\int_{-h}^{h}\mathrm{d}x_2[(-H_2 + \overline{H}_2)\delta(\mathbf{A}\cdot\mathbf{s}) + (\mathbf{H}\cdot\mathbf{s} - \overline{\mathbf{H}}\cdot\mathbf{s})\delta A_2].
\end{aligned}
\tag{B.1}
$$

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